Journal of Global Optimization **16:** 295–297, 2000.

Letter to the Editor

Remarks on an algorithm for reverse convex programs

H.D. TUAN
Department of Information and Control, Toyota Technological Institute, Hisakata 2-12-1, Tempaku, Nagoya 468-8511, Japan

An algorithm has been proposed in Strekalovsky and Tsvendor, 1998 for solving the global optimization problem:

$$\min f(x) \quad \text{s.t.} \quad x \in S,\ g(x) \geqslant 0 \tag{P}$$

where $f(x)$ is a continuous function, S a closed convex set in R^n, and $g(x)$ a differentiable convex function, satisfying the following assumptions:

(i) $S \subset \text{int}(\text{dom})g$;
(ii) the problem has a finite optimal solution;
(iii) no global optimal solution exists such that $g(x) > 0$;
(iv) there exists an $x \in R^n$ such that $g(x) < 0$;
(vi) for every y of the level surface $g(y) = 0$ there exists a $v = v(y) \in S$ satisfying

$$\langle g'(y), v - y\rangle > 0 \tag{1}$$

The last condition implies that 0 is a not a local maximum of $g(x)$ over S, hence the problem (P) is stable (see e.g. Horst and Tuy, 1990, Lemma X.2). From the assumptions it then follows that a necessary and sufficient condition for a feasible solution z to be optimal is

$$\{x \in S \mid f(x) \leqslant f(z)\} \subset \{x \mid g(x) \leqslant 0\}. \tag{2}$$

(see, e.g., Tuy, 1995, Proposition 8). Since the closed convex set $G := \{x \mid g(x) \leqslant 0\}$ equals the intersection of all its supporting halfspaces: $G = \cap_{g(y)=0}\{x \mid \langle g'(y), x - y\rangle \leqslant 0\}$, condition (2) is equivalent to requiring that

$$\sup_{y\in\partial G}\ \sup_{x\in D_z} \langle g'(y), x - y\rangle \leqslant 0, \tag{3}$$

where $\partial G = \{y \mid g(y) = 0\}$, $D_z := \{x \in S \mid f(x) \leqslant f(z)\}$.

Based on this optimality criterion (condition (E2)), the algorithm proposed in Strekalovsky and Tsvendor, 1998 for solving (P), consists basically in the following.

Let $z \in \partial G$ be a feasible solution which is also a stationary point computed by any local optimization procedure. Solve the problem

$$\gamma(z) := \sup_{y \in \partial G} \sup_{x \in D_z} \langle g'(y), x - y \rangle. \tag{4}$$

If $\gamma(z) \leqslant 0$ then z is optimal solution of (P). Otherwise, let $(w, u) \in \partial G \times D_z$ satisfy $\langle g'(w), u - w \rangle > 0$. Then $u \in S$, $f(u) \leqslant f(z)$, $g(u) > g(w) = 0$, so starting from u one can compute a new stationary point $z' \in \partial G$ better than z and the cycle can be repeated from z' in place of z.

The crucial point in this scheme is how to solve (4). In Strekalovsky and Tsvendor, 1998 the following procedure is proposed:

(1) Select a grid $R = \{y^1, \ldots, y^N\} \subset \partial G$.

(2) For each $i = 1, \ldots, N$ solve

$$\sup_{x \in D_z} \langle g'(y^i), x \rangle \tag{PL_i}$$

obtaining an optimal solution $u^i \in D_z$, then solve

$$\sup_{y \in \partial G} \langle g'(y), u^i - y \rangle \tag{LV_i}$$

obtaining an optimal solution $w^i \in \partial G$.

(3) Compute $\eta := \max_{i=1,\ldots,N} \langle g'(w^i), u^i - w^i \rangle$. If $\eta > 0$, then (w, u) has been found such that $\langle g'(w), u - w \rangle > 0$.

(4) If $\eta \leqslant 0$, accept z as an optimal solution of (P).

(To make the basic idea clearer we suppose that every subproblem involved is solved exactly. In the actual algorithm stationary points are computed with tolerance ε and the subproblems in 2) are solved with tolerance δ, so in 4) optimality is interpreted with tolerance (ε, δ)).

In Strekalovsky and Tsvendor, 1998 it is reported that the above algorithm has solved successfully problems of dimension up to 400. Unfortunately, all the test problems used are obtained merely from an extremely easy problem (P1) in R^n by considering different values of n. A global optimal solution of (P1) in whatever dimension n can be readily found by simple computations or can be easily computed using other known algorithms. Therefore the fact that the algorithm can solve these problems with dimension $n = 400$ or higher does not say anything about its efficiency. Furthermore, the question of how to construct a suitable set R for problems whose global optimal solution is known in advance does not present any interest.

In fact, the algorithm suffers from several serious difficulties:

(i) The “linearized" problems (PL_i) and the “level" problems (LV_i), $i = 1, \ldots, N$ are nonconvex problems very hard to solve, if $f(x)$ is nonconvex or/and $g(x)$ is nonquadratic. For these problems no algorithm is currently available, so the algorithm reduces the original problem (P) to a sequence of subproblems which, in principle, are still very difficult, unless $f(x)$ is convex *and* $g(x)$ is quadratic.

(ii) Even if (PL_i) and (LV_i) can be solved efficiently (e.g. when $f(x)$ is convex and $g(x)$ is convex quadratic, as in all computational experiments reported in the paper), there is no guarantee that the accepted solution is correct. Specifically, since R is a grid of ∂G, we have

$$\max_{y\in R}\max_{x\in D_z}\langle g'(y), u-y\rangle \leqslant \eta := \max_{i=1,\dots,N}\langle g'(w^i), u^i - w^i\rangle$$
$$\leqslant \max_{y\in\partial G}\max_{x\in D_z}\langle g'(y), u-y\rangle,$$

so from $\eta \leqslant 0$ it does not generally follow that (3) is satisfied. Therefore a grid R is defined to be a “resolving set" (more precisely, a (z, ε, δ)-resolving set) if

$$\eta = \max_{i=1,\dots,N}\langle g'(w^i), u^i - w^i\rangle \leqslant 0 \tag{5}$$

implies the optimality of z, with tolerances ε, δ. However, the crucial question of how to construct a resolving set is not addressed seriously, and we must be content with such vague indications as “one must have a deep understanding of the nature and the structure of the problem and the condition (3)", etc.

Thus, the algorithm is short on theoretical foundation and when it is implementable (i.e., when $f(x)$ is convex and $g(x)$ is quadratic) there is no guarantee that the solution it provides is correct. The claim that the algorithm can solve large scale reverse convex programs of dimension up to 400 is misleading.

Finally, we note that the problem (3) is no easier than (2). When $f(x)$ is convex, and $g(x)$ is convex but nonquadratic, (2) is a convex maximization problem, solvable by currently available algorithms, while (3) is a difficult nonconvex problem.

References

Strekalovsky, A.S. and Tsvendor, I. (1998) Testing the $\mathfrak{R}$-strategy for a reverse convex program, *Journal of Global Optimization* 13, 61–74.

Horst R. and Tuy H. (1990) Global Optimization: Deterministic Approaches, (1st edn) Springer.

Tuy H. (1995) D.C. optimization: theory, methods and algorithms in (eds.), *Handbook of Global Optimization* Horst, R. and Pardalos, P., Kluwer.